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ON THE NORMALIZER PROBLEM FOR INTEGRAL GROUP RINGS OF TORSION GROUPS

BY

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ABSTRACT

In this paper, we investigate the normalizer property for the integral group ring of a torsion group. We show that this property holds for locally finite nilpotent groups. A necessary and sufficient condition for this property to hold for any torsion group is also given.

1. Introduction and preliminary

Let G be a group and $\mathcal{U}(\mathbb{Z}G)$ be the group of units of the integral group ring $\mathbb{Z}G$ of a group G. The problem of investigating the normalizer $N_{\mathcal{U}}(G)$ of G in $\mathcal{U}(\mathbb{Z}G)$ has been already studied by several authors and is related to some central problems in the theory of group rings (see [7, 16] for detail). Clearly, $N_{\mathcal{U}}(G)$ contains G and also contains $\mathcal{Z} = \mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$, the subgroup of central units of $\mathcal{U}$.

Problem 43 in [16] asks whether $N_{\mathcal{U}}(G) = G\mathcal{Z}$ when G is finite. The equality was first shown to hold for finite nilpotent groups by Coleman [3], and later extended by Jackowski and Marciniak [5] to all finite groups having a normal Sylow 2-subgroup. In particular, this property holds for all finite groups of odd order. We remark that there is a close relation between this question and the isomorphism problem (see Mazur [13, 14, 15]). Hertweck first found counterexamples

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to the normalizer problem, and then, using them and a smart generalization of Mazur's results, he managed to construct a counterexample to the isomorphism problem ([4]).

Recently, a certain amount of work on this topic has been done. Parmenter, Sehgal and the author [11] proved that the normalizer property holds for any finite group G, such that $R(G)$ is not trivial, where $R(G)$ denotes the intersection of all nonnormal subgroups of G. This has an important application in studying the hypercentral units in integral group rings (see [1, 2, 9, 10]). In the meanwhile, Marciniak and Roggenkamp [12] showed that this property holds for finite metabelian groups with an abelian Sylow 2-subgroup. The latter has been extended by the author [8]. In that paper, we first gave a necessary and sufficient condition for the normalizer property to hold for the integral group ring of a finite metabelian group. We then confirmed that the property holds for several types of finite metabelian groups in which a Sylow 2-subgroup is not necessarily an abelian group. For instance, the normalizer property holds for the integral group ring of a split finite metabelian group with a dihedral Sylow 2-subgroup. Little is known about this property when the group basis G is a torsion group. In this note, we first show that the property holds for locally finite nilpotent groups (Theorem 2.2). We then extend a result of Jackowski and Marciniak to arbitrary torsion groups (Theorem 2.4).

Next we introduce some terminology and preliminary results.

Definition 1.1: Let G be a torsion group. A subgroup P is called a Sylow p-subgroup of G for a prime number p, if P is a maximal p-subgroup of G.

It is not hard to see that there exists a maximal p-subgroup of G by Zorn's Lemma. We remark that Sylow theorems for finite groups are no longer true in this context. For example, not all Sylow p-subgroups are conjugates of one another. We need the following result, and its proof can be found in [6] (1.B.10 Proposition).

LEMMA 1.2: *Let G be a locally finite nilpotent group. Then $G = \sum O_p$, where O_p is the normal maximal p-subgroup of G, and the direct sum is taken over all primes p.*

Every unit $u \in N_{\mathcal{U}}(G)$ induces an automorphism φ of G such that $\varphi_u(g) = ugu^{-1}$ for all $g \in G$. We now consider the subgroup $\mathrm{Aut}_{\mathcal{U}}(G)$ formed by all such automorphisms and it is not hard to see that the normalizer problem described in [16] is equivalent to the Question 3.7 in Jackowski and Marciniak [5]:

"Is $\mathrm{Aut}_{\mathcal{U}}(G) = \mathrm{Inn}(G)$ for all finite groups?"

It is convenient to use this equivalent form to discuss the normalizer problem here and our notation follows that in [16].

2. The normalizer $N_{\mathcal{U}}(G)$ for nilpotent groups

In this section, we first confirm that the normalizer property holds for all locally finite nilpotent groups, which extends Coleman's result. Then we give a necessary and sufficient condition for this property to hold for any torsion group. We need the following lemma, which is a special case of Theorem 9 of [15].

LEMMA 2.1: *Let G be a torsion group and P be any p-subgroup of G. For any $u \in N_{\mathcal{U}}(G)$, define $\varphi_u \in \mathrm{Aut}(G)$ such that $\varphi_u(g) = ugu^{-1}$ for every $g \in G$ as before. Then restricted to the subgroup P, the automorphism φ_u becomes inner. Moreover, we have $\varphi_u|_P = \mathrm{conj}(x_0)|_P$ for some $x_0 \in \mathrm{supp}(u) \subset G$. In particular, if G is a p-group, then $\mathrm{Aut}_{\mathcal{U}}(G) = \mathrm{Inn}(G)$, so the normalizer property holds for G.*

We include a proof for completeness.

Proof: Let $u = \sum u(x)x \in N_{\mathcal{U}}(G)$, where $u(x) \in \mathbb{Z}$ and $x \in \mathrm{supp}(u)$. For every group element $g \in G$, $\varphi(g) = ugu^{-1}$ is also a group element. Rewrite $u = \varphi(g)ug^{-1}$, and hence $\sum u(x)x = \sum u(x)\varphi(g)xg^{-1}(*)$. This forces that $\varphi(g)xg^{-1}$ is in the support of u for all $g \in G$. Define a left group action σ_g of G on $\mathrm{supp}(u)$ as follows: $\sigma_g(x) = \varphi(g)xg^{-1}$. It follows from $(*)$ that $u(x)$ is a constant on each orbit of x. Restricting the action to P, we have that the p-subgroup P acts on $\mathrm{supp}(u)$, and thus every orbit must have a length of p-power. It follows that

$$\pm 1 = \epsilon(u) = \sum c_i p^{l_i},$$

where ϵ is the augmentation map, p^{l_i} is the length of the orbit of x_i and $u(x_i) = c_i$. This forces that $p^{l_j} = 1$ for some j; that is to say there is a fixed point of this action, say x_0. Therefore, we have $\varphi(g)x_0g^{-1} = \sigma_g(x_0) = x_0$ for all $g \in P$. Consequently, $\varphi(g) = x_0gx_0^{-1}$, and thus $\varphi|_P = \mathrm{conj}(x_0)|_P$. We are done. ∎

Now we show that the normalizer property holds for locally finite nilpotent groups, which extends Coleman's result [3].

THEOREM 2.2: *Let G be a locally finite nilpotent group. Then the normalizer property holds for G.*

Proof: For any $u \in N_{\mathcal{U}}(G)$, define φ such that $\varphi(g) = ugu^{-1}$ as before. Since φ^2 is inner by Proposition (9.5) of [16], if some odd power of φ is inner, then φ is inner too and we are done. We note that it follows from Theorem 1 of [15] that $\mathrm{Aut}_{\mathcal{U}}(G)$ is a torsion group for any torsion group G. By taking a suitable odd power of φ, we may assume that the order of φ is a power of 2. It follows from Lemma 1.2 that $G = \sum O_p$, where O_p is the largest normal p-subgroup of G and the direct sum is taken over all primes p. By Lemma 2.1, we have that $\varphi|_{O_p} = \mathrm{conj}(x_p)|_{O_p}$, where $x_p \in \mathrm{supp}(u)$. Since $\mathrm{supp}(u)$ is a finite set, we can choose a large odd integer l such that $(x_p)^l$ are 2-elements for all $x_p \in \mathrm{supp}(u)$. Again by taking a suitable odd power of φ, we may assume that all of these x_p are 2-elements. Therefore, $x_p \in O_2$, and this gives that $\varphi|_{O_p} = \mathrm{conj}(x_p)|_{O_p} = \mathrm{id}|_{O_p}$ for $p \neq 2$ since x_p commutes with every element of O_p. We claim that $\varphi = \mathrm{conj}(x_2)$. To see this, we note that $x_2 \in O_2$, so $\mathrm{conj}(x_2)|_{O_p} = \mathrm{id}|_{O_p} = \varphi|_{O_p}$ for $p \neq 2$ and $\mathrm{conj}(x_2)|_{O_2} = \varphi|_{O_2}$. We are done. ∎

In the remaining part, we extend Jackowski and Marciniak's result ([5], 3.5 Theorem) to arbitrary torsion groups. We need the following result and a proof can be found in [5] or [16].

LEMMA 2.3: *Let G be an arbitrary group and let u be a unit of $\mathbb{Z}G$. Then $u \in N_{\mathcal{U}}(G)$ if and only if $uu^* \in \mathcal{Z}(\mathbb{Z}G)$.*

For a fixed p-subgroup P of G, denote by I_P the set of all involutions in $\mathrm{Aut}_{\mathcal{U}}(G)$ which keep P pointwise fixed:

$$I_P = \{\varphi \in \mathrm{Aut}_{\mathcal{U}}(G) | \varphi^2 = \mathrm{id} \text{ and } \varphi|_P = \mathrm{id}\}.$$

THEOREM 2.4: *Let G be any torsion group. If $I_P \subseteq \mathrm{Inn}(G)$ for a maximal Sylow 2-subgroup P of G, then $\mathrm{Aut}_{\mathcal{U}}(G) = \mathrm{Inn}(G)$.*

Proof: Let $u \in N_{\mathcal{U}}(G)$ and let $\varphi \in \mathrm{Aut}_{\mathcal{U}}(G)$ be the normalized automorphism induced by u as before. It follows from Lemma 2.1 that $\varphi|_P = \mathrm{conj}(g_0)|_P$ for some group element $g_0 \in \mathrm{supp}(u)$. Conjugating φ by a group element if necessary, we may assume that $\varphi|_P = \mathrm{id}|_P$. Let $v = u^* u^{-1}$. Then by Lemma 2.3, we have

$$vv^* = (u^* u^{-1})((u^{-1})^* u) = u^* (u^* u)^{-1} u = u^* (uu^*)^{-1} u = 1.$$

Hence v is a trivial unit and then $v = t$ for some group element $t \in G$. This says that $u^* = tu$, and moreover, $\varphi^2 = \mathrm{conj}(t^{-1})$. As mentioned earlier in the proof of Theorem 2.2, we may assume that the order of φ is a power of 2. Furthermore, by the same reason, we may assume that t is a 2-element. Since $\varphi^2|_P = \mathrm{id}|_P$, we have $t \in C_G(P)$ the centralizer of P in G. Note also that t is a 2-element and P is a maximal Sylow 2-subgroup, so we conclude that $t \in \mathcal{Z}(P)$ the center of

P. As we mentioned earlier in the proof of Lemma 2.1, we can define a group action from P to $\mathrm{supp}(u)$. Write $u = \sum u(x)x$ as before. We recall that under this group action $u(x)$ is constant on each orbit of x and the length of each orbit is always a power of 2 in the present case. Moreover, the length of the orbit of x is 1 if and only if $x \in C_G(P)$. Rewrite $u = \beta_0 + \beta_1$, where $\mathrm{supp}(\beta_0) \subseteq C_G(P)$ and $\mathrm{supp}(\beta_1) \subseteq G \backslash C_G(P)$. Taking the augmentation of u, we obtain

$$\pm 1 = \epsilon(u) = \epsilon(\beta_0) + \sum c_i 2^{k_i} \quad \text{where } k_i \geq 1.$$

Hence $\epsilon(\beta_0)$ is an odd number. It follows from the identity $u^* = tu$ that $\beta_0^* = t\beta_0$. Let $\beta_0 = \sum \gamma_h h$. Then we have $\sum \gamma_h h^{-1} = \sum \gamma_h th$ or $\sum \gamma_h h = \sum \gamma_h h^{-1} t^{-1}$. Thus $\gamma_h = \gamma_{h^{-1}t^{-1}}$ for all $h \in \mathrm{supp}(\beta_0)$. Since $(h^{-1}t^{-1})^{-1}t^{-1} = h$, this contradicts that $\epsilon(\beta_0)$ is an odd number unless $h = h^{-1}t^{-1}$ for some $h \in \mathrm{supp}(\beta_0)$. We now conclude that $t^{-1} = h^2$ for some 2-element $h \in C_G(P)$, and hence, $h \in \mathcal{Z}(P)$. Define an inner automorphism $\rho = \mathrm{conj}(h^{-1})$. It follows that $\rho\varphi|_P = \mathrm{id}|_P$ and $(\rho\varphi)^2 = \rho^2\varphi^2 = \mathrm{conj}(t)\,\mathrm{conj}(t^{-1}) = \mathrm{id}$, so $\rho\varphi \in I_P$. Consequently, $\rho\varphi$ is inner and thus φ is inner as desired. ∎

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